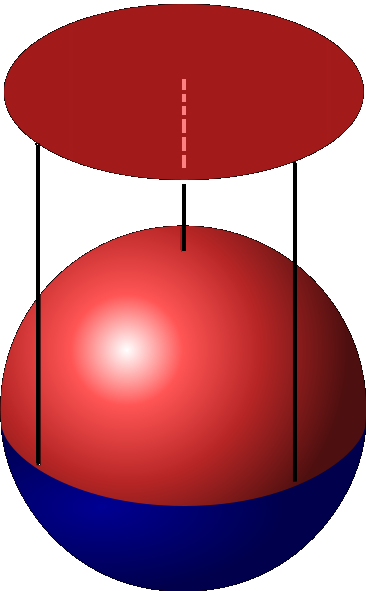
**Tensors and stuff**

(note we’re using Einstein summation convention, implicitly; maybe see Appendix at bottom)

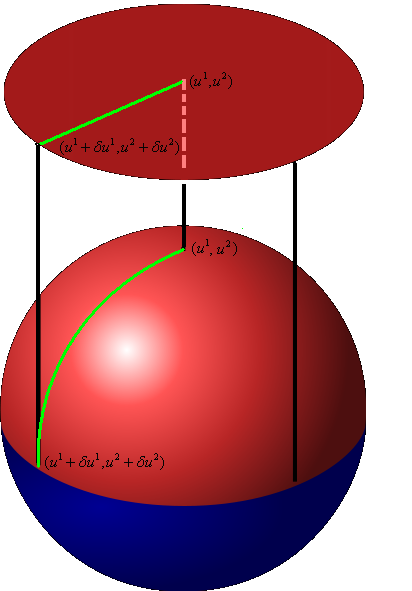
**1. Manifolds and Metrics**

In order to describe the points in space, one draws up a coordinate system and assigns a set of coordinates (numbers) to each point in that space. This set of coordinates is called a manifold. For instance, the longitude/latitude numbers constitute a manifold describing positions on the Earth. x,y,z coordinates describing 3D Euclidean space constitute a manifold. So do r,θ,φ spherical coordinates. Going to higher dimensionality, ct,x,y,z, are coordinates specifying points in space-time. But a manifold is just a set of numbers (coordinates). And it doesn’t contain any information about the structure of the space. For instance we can use the same set of numbers to describe a disk surface as a hemisphere surface via a simple one-to-one projection.



So in order to describe the particular surface (or hyper-surface) that we’re interested in we need one more thing – a metric which contains information about the curvature of the space.

The thing which distinguishes the disk geometry from the hemisphere geometry is the curvature of the surfaces. Or alternatively, the distinction is that when we go from (u1, u2) to (u1 + δu1, u2 + δu2) on the disk we travel a different distance from when we go from (u1, u2) to (u1 + δu1, u2 + δu2) on the hemisphere, as illustrated below:

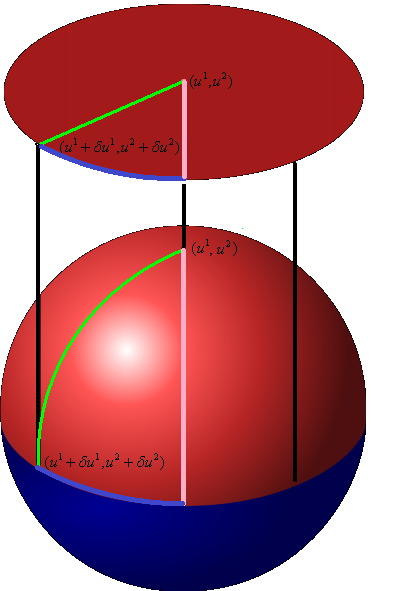


The metric is the entity which supplies information about the geometry of the surface. Let’s calculate the distance (squared) between two nearby points uα and uα + duα (okay they’re not nearby, but exaggerated for clarity). Generally speaking this will be some function of products of differential displacement in coordinates. Note I believe it’s a general rule that for *actually* tiny *and* orthogonal displacements along the hypersurface, we can use the Pythagorean theorem so that ds2 = (displacement1)2 + (displacement2)2, etc. In any event, we implicitly invoke this equality every time we calculate the metric. This is saying that any surface is at least *locally* flat. In any particular geometry, it will look like,



So we can see that the distance between two nearby points is fundamentally connected to the metric, gαβ. And this gives us an explicit way to calculate the metric. We simply vary our coordinate by a differential amounts and use a ruler to determine how far we went. Note this is explicitly measurable for us, in the space itself. We don’t have to stand outside of it. From our use of the Pythagorean theorem above, it is implicit that any geometry may be described *locally* by a unit metric – all we have to do is choose a coordinate system characterized by mutually orthogonal displacements, and scale the coordinates so that the coordinate measures distance itself. And in any event, since gαβ is symmetric, it can be diagonalized by an appropriate change of base. And so we come to the same conclusion. This fact will be useful later.

It troubles me that we can apparently not tell if our space is curved by making local measurements, since Pythagorus’s theorem will always hold for small displacements, i.e., ds2 = dx2 + dy2 + dz2 for sure, for us in our space or any 3D space. But I guess we can tell if our space is curved by making *finite* mutually orthogonal displacements, like below:



and if Pythagorus’s theorem doesn’t hold, i.e., if it isn’t true that in some 3D space we have Δs2 = Δx2 + Δy2 + Δz2, then we know that there is curvature in that space. We can formally calculate distances over finite displacements using the metric again. Just parameterize the path as uα(t) and then calculate:



Now let’s do quick discussion of the volume element. When doing volume integrals, we have to integrate over the volume element. Consider a metric:



and then we can write dV as:

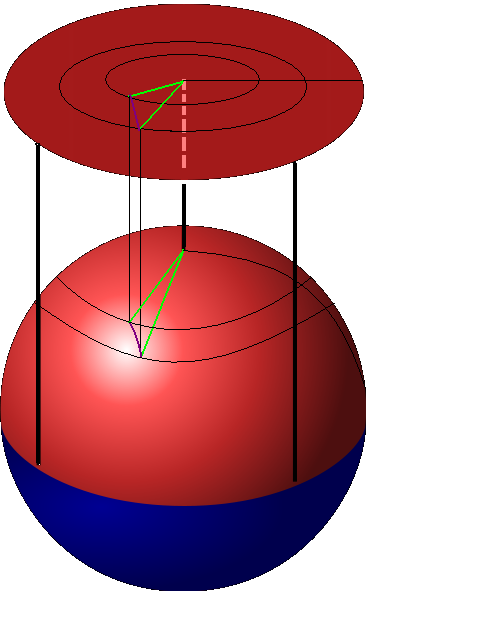


Note that this is completely general, in a sense, because any manifold can be made diagonal by a change of coordinates. And so more generally we can write this as:



**Example**

Let’s do a radial manifold describing, alternately, a disk and hemisphere geometries…



For the disk we have:



and consequently a metric:



and for the hemisphere we have:



where R is the radius of the hemisphere. And consequently we have the following metric:



Notice the singularity in the metric when r = R, which indicates that the distance traversed when moving a little distance dr when r is close to R is near infinity. This makes sense because of the nearly vertical slope of the surface when r ≈ R. So the message to be taken from this is that coordinates (manifold) do not describe a geometry until a metric is supplied (which gives information about how the space is curved – or not – at each point).

Now let’s look at the distance along a path. Let’s say our path goes from r = 0 to R/2 along constant angle φ1 between time (0,1). Then it stays at r = R/2, but φ changes from φ1 to φ2 between time (1,2). Then φ2 remains constant and r changes from R/2 to 0 between time (2,3):



What is the distance traversed for the disk metric? And what is the distance traversed for the hemisphere metric?

Well, in the disk case, we have:



So,



and thus,



And in the hemisphere case, we have (note I’m taking the absolute value of the dr’s):



We’ll recognize the antiderivative as sin-1(t). So we have:

